On Teaching Physics and Mathematics

by Hermann von Baravalle, PhD

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Translator's Note

In translating this paper small inaccuracies and printing errors in the German original, as published by Waldorfschul-Spielzeug und Verlag, Stuttgart, 1928, have been corrected. Other items requiring comment or clarification, identified by numbers in the text, follow the text as Translator's Notes. All footnotes in the text are the author's.

This translation has profited greatly from discussions with John Hoffman and David Booth, teachers at the Green Meadow Waldorf School in Spring Valley, New York, and from the sensitivity to language of Ruth Pusch in editing the final text.

Herbert Winter

Publisher's Preface

It has been for me a very happy undertaking to republish Hermann von Baravalle's inaugural dissertation on behalf of the Mathematical-Astronomical Section at the Goetheanum. Those of us who have a genuine feeling for the forces of growth and development, acquired through patient study of Rudolf Steiner's spiritual scientific ideas indicated in *Knowledge of Higher Worlds and its Attainment,* will be delighted to meet this book again after seven years—and be stirred by it. The ideas of this highly gifted teacher and artist that we know from his later work, especially his *Geometrie in Bildern* and *Durchblick durch die Erde,* are already present in this one, as a future plant is contained in its seed. His books have brought new life and creativity to the otherwise dry mathematics instruction; this first pedagogical work of his is filled with the vibrant experience of his own teaching practice. Every page bespeaks the love for his young students and their problems, a love which, fructified by Rudolf Steiner's educational ideas, brought the seed to flowering. The suggestions given in the most diverse domains of physics, mathematics, and astronomy present still today a beautiful introduction to the methods of teaching these subjects.

Elisabeth Vreede PhD
For the Mathematical-Astronomical Section
of the School of Spiritual Science,
Dornach 1928

Author's Preface

(for the first publication of the work in the series *Wissenschaft und Zukunft* 1921)

Revising the school system today is not merely a problem for the experts but has become a problem of general interest. On all sides there are attempts to remove irrelevant material from the curriculum. The following work is meant to be a contribution to this endeavor. Since I decided from my earliest youth to dedicate my life to teaching, I tried during my own schooling to make observations on myself and my friends that could be useful for my future profession. Later, too, I constantly considered these problems and tried to find out where in the course of instruction students have the greatest difficulties. The actual examples that I met in the course of time have helped me in my attempt to meet those difficulties. What started as instinctive striving became a goal-oriented task when I became acquainted with the pedagogical ideas of Dr. Rudolf Steiner, now being used and developed with great success in the Free Waldorf School in Stuttgart for students of elementary and high school age, and in the Goetheanum in Dornach near Basel in Switzerland for those of college age.

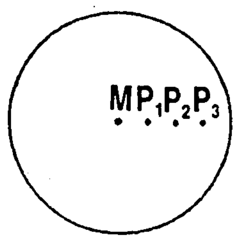
Hermann von Baravalle

On Teaching Physics

During the first years of physics instruction, and even later, we find again and again that students have the greatest difficulties understanding the *concepts* of physics. The teacher, of course, has spent considerable time working in the field. Things that seem obvious to him, for instance, the definition of speed under uniform motion as the distance covered in unit time, are often extremely difficult for students to understand, therefore often simply memorized by them. If we look for the cause, we have to admit: the students knew what speed was before starting the study of physics, but they knew it as experienced when riding in a car or a train. Surely every child has had the happy experience of enjoying high speed. Now, it is only necessary to trace the history of physics to realize how many thousands of years it took to get from the mere experience of speed to its measurement in terms of distance and time. The necessary progression to exact numerical measurements may perhaps be presented as follows: The experience of speed as such is not measurable, therefore does not admit attachment of a numerical measure. If one wants to introduce such a measure one must turn to measurable quantities which are connected with bodies in motion, and which will permit a unique determination of speed. Now, the measurable quantities connected with motion are the distance covered and the time elapsed. One of these quantities alone is not sufficient for a unique determination of speed; both together are sufficient. But, in order not to need two measures to determine speed, one for distance and one for time, one lets only one of these quantities vary, and sets the other equal to a constant, namely unity. Now, logically no objection could be raised if one were to define speed in such a way that the distance is set equal to unity, so that speed would be defined as the time necessary to cover a unit distance. But this would cause motion experienced as faster to be represented by a smaller numerical measure of speed; it is therefore more reasonable to set the time equal to unity, and to define speed as the distance covered in unit time. The teacher, in introducing a definition, must be aware of the whole path necessary to arrive at that definition; only then can he understand the difficulties students will have when this path is simply skipped, and why he must try to present everything necessary to let the student take part in this development. Here, attention to the history of physics is one of the best methods, just as it is always of great educational value to point to the great achievements of a Galileo, Kepler, etc., and to the difficulties they had to overcome.

We often find in teaching that we add new ideas to those introduced earlier without first carefully developing the reason why the earlier ideas are no longer sufficient. We find this, for instance, when we proceed from linear to rotational motion, and substitute the ideas of angular velocity, angular acceleration, torque, etc., for velocity, acceleration, force, etc. If we simply bring new definitions, the students may have a slight feeling of arbitrariness in the introduction of these ideas, just as everything that is not understood must appear arbitrary. Here it is therefore necessary to show first how on a rotating disk all points do not have the same velocity, that one point (*M* in Fig. 1) remains completely at rest, while the others move with greater velocity the farther they are from *M*, so that the simple idea of velocity is no longer sufficient. One might now say that in order to have a unique measure for the angular velocity of a disk one could choose certain points (those with $r = 1$) and use their velocity to represent the velocity of the disk. In this way one gets directly to the idea of angular velocity, and from there to its measurement in terms of revolutions per unit time (e.g., rpm).

Fig. 1



An objection to this method is that the ideas developed for linear motion are simply modified to be used for angular motion. It is better not to teach the student to solve everything with ideas developed earlier, because such laziness in thinking has already created much mischief in science (the theory of light emission, etc.). Instead, the student should be led from the beginning to comprehend the characteristics of each new phenomenon through new ideas. In linear motion it turned out that speed could be measured by the number of distance units traversed in a unit of time, where the units were arbitrary. It turns out that in rotational motion there is a natural unit, namely a complete rotation, the return to the original position, so that the number of complete rotations in a unit of time appears as the natural measure of angular velocity. This is also the measure used most often in practice (rpm of an electric motor, a steam engine, a propeller, etc.).

Only now can we arrive at the angular velocity and say: If in certain problems (e.g., problems of centrifugal force) what matters is not the rotating disk as a whole, but the velocity reached by certain points on the disk, one must find simple ways to compute these from the rotational velocity (rpm). Here, one only needs to come back to the measures defined for linear motion, distance and time. For a point at a distance r from the center of rotation, the distance covered in one rotation is $2\pi r$. In one unit of time there are n rotations, so the distance covered in a unit of time is $2\pi rn$. The velocity increases in proportion to the magnitude r. If one has to compute the velocities for several points one need only multiply the quantity $2\pi n$, representing the velocity for the points with $r = 1$, by r. This velocity can therefore always be chosen as the basic velocity (angular velocity). It is now self-evident that one can also find the angular velocity ω from the velocity v of any point at a distance r from the center of rotation by dividing v by $r : \omega = v/r$. Only when physical concepts are introduced clearly and exactly can the students be led to a real understanding.

For this it is necessary to differentiate strictly between those formulas of physics that represent an introduction of new concepts, and those which contain a law of nature connecting such concepts. So, for instance, the formula $P = mv^2/r$ (P = centrifugal force, m = mass, v = velocity of a point at distance r from the center of rotation) represents a law of nature; by no means, however, does $F = ma$. Yet, how often does one find this formula presented in instruction as if it were a law of nature! This formula introduces the measure for force, which we might perhaps present to students in the following way, assuming that the concepts of velocity, acceleration, inertia, etc. are already familiar. We explain that what students have known as force is again not accessible to measurement. A force becomes measurable only when it has set a body in motion, so that one can measure an acceleration of that body. If the force would provide the same acceleration to all bodies, the acceleration would immediately provide a measure for force. This, however, is not the case. If we nevertheless want to retain the acceleration as a measure we have to agree on a body to be put in motion, so that we may define as a measure of force the acceleration given to that particular body. But sometimes this body may not be available; one would therefore like to be able to compute the acceleration of the unit body from the acceleration of any available body. Now it turns out that a body which, on application of a small force, moves just half as fast as the unit body, will also, on applying a large force, have half the acceleration of the unit body, so that the conversion is achieved simply by multiplying by 2. In fact, for every body a number can be found which indicates how much greater the acceleration of the unit body would be on application of the same force. This number is called the mass = m of the body, and so the measure of force can be taken as $F = ma$..

Likewise, Gay-Lussac's Law must not be presented as a law of nature. It is an immediate consequence of the introduction of the concept of temperature. If we decide to measure temperature by the expansion of a body, and if we divide the expansion between freezing point and boiling point of water into equal parts and extrapolate, then necessarily $l_t = l_0\,(1 + \alpha t)$. On the other hand, Mariott's Law, $PV = k$, recognizes a new concept.

Poorly understood symbols can be just as detrimental as poorly understood definitions or formulas. How many students know where the symbols for degree (°), minute (′) and second (″) come from? From astronomical considerations, one took the 360th part of the full circle as a unit, and then constructed divisions. The first division divided the degree into 60 parts, and the result of this first reduction was called "minute" (*minutus* = reduced). The second division led to the "second" (*secundus* = the second). Accordingly the minute as first reduction is denoted by ′, the second as a second reduction by ″, and so the degree as zeroth reduction by °.

The rule to keep any poorly understood material out of the lesson requires unlimited honesty in the field of experiments. There is nothing more harmful than to pretend, after an unsuccessful or partly successful experiment, that all is in order and that the result has clearly established the theory. It is obvious that we must make every effort to perform experiments successfully, and that it is better not to perform an experiment at all rather than to have it miscarry. Nevertheless, it can happen that an experiment does not come out as expected. In such an event, we must honestly admit to failure, determine its possible cause, and repeat the experiment next time.

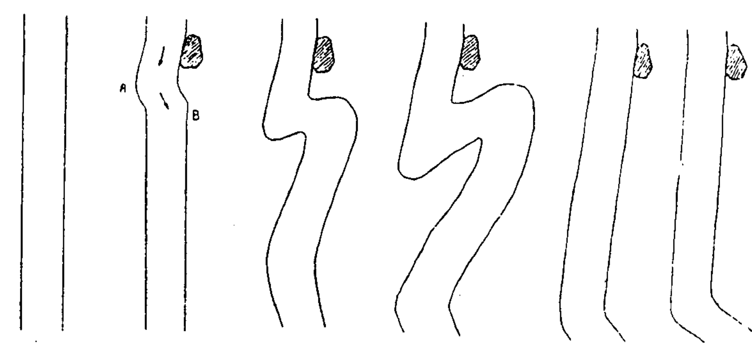
Instead of demonstrating effects by means of small-scale experiments it is often much better to show their monumental results in nature. For instance, the effect of temperature on volume can be shown most clearly by taking the students to a quarry where the rock layers are nearly vertical or show folds. We then show by fossilized sea shells, etc., that these layers had once been laid down horizontally by water; later they moved in such a grandiose manner because of the earth's shrinkage due to the decrease in temperature.[1] At the same quarry we can explain that the effects of expansion and contraction through changes in temperature are the primary cause of the weathering of the rocks. Even though these expansions in solid bodies are small and all kinds of instrumentation are necessary to make them visible, they occur with such tremendous force that they can even shatter the rocks.

Through such procedures we can make the students realize that experiments, such as these concerning expansion, are not special marvels happening only in the classroom, but that they serve more than anything to bring before our eyes things that occur everywhere in the outside world. We can reach this goal very successfully by letting the students themselves discover where phenomena observed in the classroom are found outside. If, for instance, after discussing the convex mirror we let the students look for convex mirrors elsewhere, we are frequently surprised by everything they come up with: china and kitchenware of all kinds, bottles, eyeglasses, spoons, patent leather shoes, ink spots, etc., right up to the human eye in which one can also see one's own mirror image.

Connection with everyday life can almost always be demonstrated, even with quite abstract mechanical concepts. Consider, for instance, the difference between stable and unstable equilibrium. The basic difference is that in unstable equilibrium a small displacement from the position of rest will result in an increased displacement, while in stable equilibrium the tendency is to return to the rest position. For this we can look for phenomena in various areas which, once they get started, tend to enlarge, for example, damage to a house, or we can look at the phenomena of daily traffic; this children themselves can observe. If streetcars travel at constant intervals, but one car is delayed for some reason, because perhaps a school lets out and many children get on, the distance to the preceding car increases, and the distance to the following car decreases. Therefore, at every stop more people will collect for this car, and it will get farther and farther behind schedule; on the other hand, the next car will find fewer people and will advance and get closer and closer to the delayed car; the even distribution of streetcars is therefore also in some kind of unstable equilibrium.[2]

Another example from outdoors would be the formation of meanders in a stream. A stream flowing in a valley in a straight bed will always carry along small rocks and will thus deepen its bed. If during this process of digging it meets up with a large rock which is toward one side of its bed, the flow will be directed toward the other side. The water runs obliquely with respect to the direction of the bed and bounds against the bank at *A* (Fig. 3). From there, it runs back into the old bed and hits the other bank at *B*. At *A* and *B*, therefore, the banks get washed away more and more, and the meander forms. The straight flow of the stream is therefore also in unstable equilibrium. This phenomenon becomes even more interesting when one considers that the meander eventually becomes so extreme that the stream breaks through and is led to the path of Fig. 6, causing new meanders (Figs. 6 & 7). The appearance of the meander, caused by every small influence, its disappearance and reappearance somewhere else, is the only explanation for the widening of the valley.[3]

Fig 2 Fig. 3 Fig. 4 Fig. 5 Fig. 6 Fig. 7

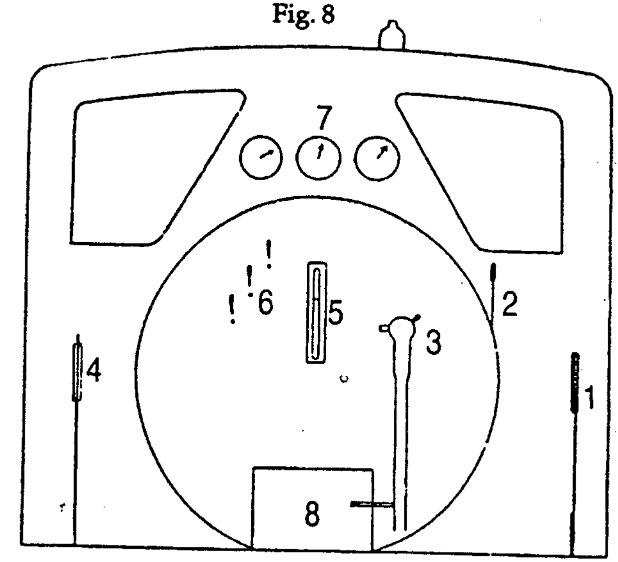


The more examples one brings from all kinds of areas to show the students how much they can learn from things they pass apathetically every day, the more likely it is that one may achieve a most important goal of scientific instruction: to stimulate the students to go through life with their eyes open. Goethe is a shining example of this. Reading his *Italian Journey*, we are amazed at how much he notices, what he observes everywhere he goes, whether it is minerals or plants or works of art. We perceive the same spirit in his *Theory of Color*. Goethe's method of research is not such that after he puts away his prisms he happily puts away his thoughts on physics too: he is a researcher wherever he is. While travelling in his carriage he observes interference phenomena on the window panes. Every cloud formation prompts him to look at the colors, etc. etc. Instead of making fun of possible inexactness in his work we should find it much more important to absorb his scientific impulses.

How much more interested the students become when we try to connect the school work with real life! It is hardly enough to study the steam engine and then to say that steam locomotives are moved by the same principle. When you do that everything remains abstract. The effect on the students is entirely different if one describes exactly where the fire box in the locomotive is placed, how the steam pipes run inside the boiler, how the steam is collected in the dome, where the water pipe runs to the steam chamber, where the cylinder is located, which wheels are driven and which only follow along, and so forth.

What joy the students have then when one draws the interior of the locomotive cab (Fig. 8) and shows what levers there are and which of them one must work to set a locomotive into motion: how one must first set the reverse lever (which sets the gears for moving forward or backwards); how then with the regulator one allows the steam to flow to the engine, setting the locomotive in motion; how in order to stop, after shutting off the flow of steam with the regulator, one has two brakes at one's disposal: the air brake (operated by the engineer) and the hand brake (operated by the fireman). Once the students have heard all this they will look at a locomotive quite differently, recognize the various parts again and notice the differences in the various types.[4]

Fig. 8



1 - reverse lever. 2 - regulator. 3 - air brake. 4 - hand brake. 5 - water gauge. 6 - test cocks. 7 - pressure gauges. 8 - fire box.

In a similar manner one can discuss the technology of a streetcar. One can point out that the driver operates the electric current switch and the electric brake with his left hand, the hand brake with his right hand. One can discuss the different uses of the two brakes. The electric brake works by shorting out the electric motor. It is thus activated by the motion of the car itself and serves therefore to brake the speed of the *moving* car. On the other hand, if it is necessary to keep a car from moving while waiting at a streetcar stop on a slope, one has to use the hand brake which works through mechanical leverage. The electric brake would first allow the car to move and only then become activated by the motion of the car; it is therefore not useful for keeping the car from moving on a slope. After such a discussion students will certainly follow the action of the driver with much greater attention and will attempt to supplement what they have learned in school with their own experience.

The controls of an automobile might be mentioned briefly. The teacher can show the gear shift lever on the right, corresponding to the reverse lever of the locomotive. It additionally permits adjustment for different speeds, and is operated by hand. The difference between the automobile and the locomotive is that in the locomotive the motion of the wheels is directly connected to the engine, while in the automobile the engine can be started without the wheels moving. The connection between engine and wheels is regulated by the clutch, which is operated with the left foot. The right foot operates the brake, so that once the engine is started the car is set in motion and stopped using only the feet, which leaves the hands of the driver free for the steering wheel.[5] When we see how great an interest the students have in all this we will not look upon it as an unnecessary extension of the already large amount of subject matter in the curriculum, but as an answer to questions that are alive in the students and must be used to awaken their interest in the real world.

The main requirement for teachers is to be intensely interested in all this, and continuously to try to increase their own knowledge in every field. This can best be achieved by approaching people in their own practical fields and asking what they think children should learn in school about the tools used in their jobs, and in what way school did not prepare students sufficiently. One result of this might be greater public interest in school problems and a willingness to cooperate in solving them; another good result will be that teachers are able to collect more and more examples from practical life for their teaching.

Just as our method of observation is stimulated by Goethe in a general sense, Goethe's points of view in particular cases are extraordinarily fruitful for teaching. Goethe always attempts to comprehend phenomena through extreme cases. This point of view may be used, for instance, to clarify wave propagation to students, since they often have much difficulty with the concept, especially as expressed by the equation:

$c = \sqrt{\frac{\varepsilon}{\rho}}$ (c = propagation velocity of a wave, ε = modulus of elasticity, ρ = density). If a point through some means is set into periodic motion on the path A-B about its point of rest R (Fig. 9), there are two extreme cases with respect to its neighboring points:

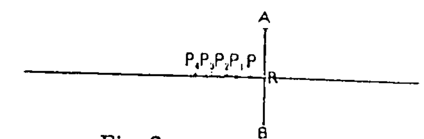


Fig. 9

1. The point has no contact with its neighboring points P_1, P_2, P_3,... Then these remain at rest independent of the motion of point P. Its motion does not propagate.

2. The point P is rigidly connected to the neighboring points P_1, P_2, P_3,..., so that they must follow its full motion. The situation then looks as shown in Figs. 10 and 11.

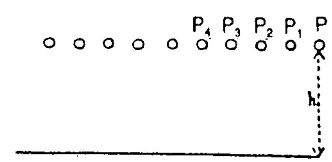


Fig. 10

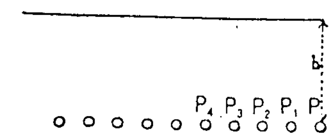


Fig. 11

The points P_1, P_2, P_3,... are raised at the same time as P by the same distance h from their point of rest, later they are lowered at the same time as P below the point of rest, like a rigid rod displaced parallel to itself. If now the connection is not completely rigid, allowing a certain elastic displacement between the several points, but if this displacement is not unrestricted, as in case 1., there will be a delay in the motion of the several points, so that after the start of the motion of P the picture of Fig. 12 develops,[6] and after several oscillations that of Fig. 13 appears.

Fig. 12

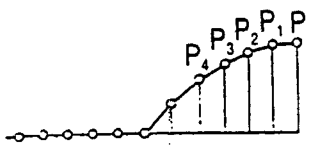
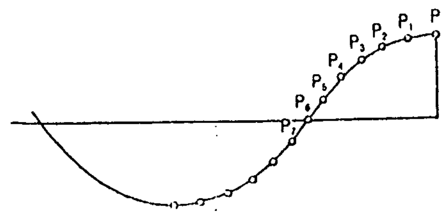
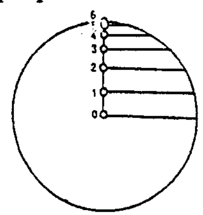


Fig. 13



The delay of the several points is greater, the greater the velocity at the corresponding displacement; but this velocity can be derived from the projection of circular motion, where the distances $\overline{01}$, $\overline{12}$, $\overline{23}$, $\overline{34}$, $\overline{45}$, $\overline{56}$ (Fig. 14), which represent these velocities graphically, can be taken as measures of these velocities. It follows that the sequential displacements are in proportion to the sine of a constantly increasing angle.

Fig. 14



The entire picture of the displacements results in a sine curve. The greater the expansion that the elastic bond permits between the several points the more will P_1 lag P, the smaller therefore will be the propagation velocity of the wave. This demonstrates the dependence of c on the modulus of elasticity, which is the essential part of the formula $c = \sqrt{\frac{\varepsilon}{\rho}}$. The modulus of elasticity is, after all, inversely proportional to the elastic displacement, since it is defined by $\lambda = \frac{1}{\varepsilon}\frac{Fl}{a}$ (λ = elastic displacement, F = force, l = length, a = cross section area).

Finally, in regard to the mathematical laws contained in physics, it is not enough simply to derive a certain formula and then

present it as the end result, but we have to awaken in the students a feeling for what is expressed in certain mathematical relationships. For instance, the formula $PV = k$ expresses a kind of equilibrium between the quantities P (pressure) and V (volume). We might present the following table:

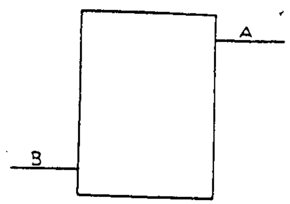
for $k = 64$

P	V
8	8
16	4
4	16
32	2
2	32
64	1
1	64

Here we can show how in doubling the pressure the volume becomes half; in quadrupling the pressure the volume at the same time decreases to one-quarter; how, on the other hand, by halving the pressure the volume doubles, etc. We can compare the relation expressed by $PV = k$ with a scale in which the falling of one pan causes the other to rise, and vice versa. The formula leads us to understand pressure and volume as two different but balanced expressions of the tendency of gases to expand (according to the kinetic theory of gases, the kinetic energy of the molecular motion in the gas) which, when not restrained, appears as an increase in volume but, when restrained, as an increase in pressure. The students must obtain an inner relation to every formula. More examples of this will be found in the section on mathematics.

In mathematical principles it is important to work out what is essential in the clearest way. Taking, for example, the principle of virtual displacement, one can hardly imagine a more beautiful presentation than the one Ernst Mach gave in his *Mechanics*. He considers a box from which two arms stick out (Fig. 15). If one is moved, the other one moves at the same time. Now, it is not at all necessary to know the mechanism enclosed in the box connecting A and B; based on the principle of virtual displacement, one can set up the equilibrium conditions from just the simultaneous motions: $F_1 ds_1 = - F_2 ds_2$, or $\sum_i F_i ds_i = 0$. This arrangement clearly shows what is important in the principle of virtual displacement, namely the simultaneously produced displacements.

Fig. 15



Instead of developing mathematical concepts purely abstractly and then applying them to physical problems, it is often much better to tie their introduction to a physical situation. This is justified historically; after all, many developments in mathematics have been stimulated by physics. Thus, instead of deriving the trigonometric functions from the right-angled triangle they can be obtained directly from the study of motion.

One can proceed in the following way. Consider the motion of a body toward the right, in the direction of the x-axis. Due to certain constraints motion only in the direction of g (Fig. 16) is possible. This includes motion to the right, but only as a part of the total motion. The relation between the two motions is a function of the angle φ, namely $\cos \varphi$. The problem might be made realistic. Suppose one wishes to reach a place which lies in the direction of the x-axis, but instead of a direct path, there is only a path in direction g. The question is then: How quickly does one progress in the direction of the x-axis? One might call the cosine the function of approach to a given direction. If, on the other hand, one is concerned with the deviation from a direction, e.g. from the direction of the x-axis; if, in other words, the question is what part of the motion is used to depart from the x-axis; one arrives at a second ratio, which again is a function of the angle φ, namely $\sin \varphi$, which one might call the function of deviation (Fig. 17).

Fig. 16

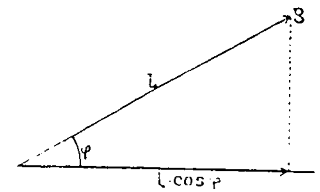
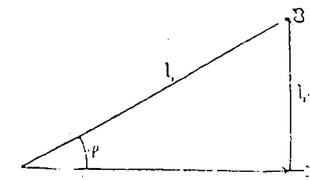
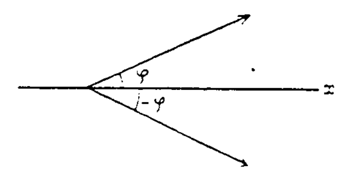


Fig. 17



The advantage of this kind of introduction of the trigonometric functions is shown especially in the derivation of further formulas, and in their applications. With this introduction, it is for instance quite obvious that $\cos \varphi = \cos(-\varphi)$, since, when one is concerned with going in the x-direction, it makes no difference whether one is on the left side or on the right (Fig. 18). On the other hand, motion in the direction of the x-axis reverses when one replaces φ with $(180^\circ - \varphi)$; hence, $\cos(180^\circ - \varphi) = -\cos \varphi$. The direction of deviation from the x-axis, however, reverses when one exchanges φ with $(-\varphi)$. If before one deviated to the left, one now deviates to the right, so that $\sin(-\varphi) = -\sin \varphi$.

Fig. 18



In physical applications one can show that the cosine appears whenever the approach of two directions is involved, while the sine appears when their deviation is involved. If a body is constrained to move in the direction g_1 (Fig. 19), but the force acts in the direction g_2, the actual motion and the work done depend on the approach of the directions; thus, $W = Fd \cos \varphi$. If, however, the rod l_1 (Fig. 20) rotates about point M, and if g_1 is the direction of an acting force, the rotation of l_1 about M depends on the deviation of the directions, because, if g_1 were in the extension of l_1, no rotation could result. Here, therefore, the sine must be used.

Fig. 19

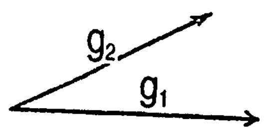
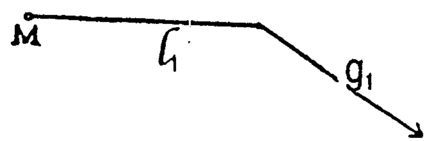


Fig. 20



To show, finally, to what extent physical points of view can help in understanding even problems of higher mathematics, we might point to the development of periodic functions from the Fourier series. How much better we will understand the importance of the sine and cosine functions among all periodic functions (according to Fourier's Theorem, all periodic functions may be represented by sums of sines and cosines) when we show how every periodic function, propagated through a resistance, by itself approaches a sine (cosine) function. For instance, the irregular diurnal or annual temperature variations, when propagated below the earth's surface, tend to approach pure sine waves.

On Teaching Astronomy

In teaching astronomy, the main thing is to base the entire development on observations, and to return again and again to what can be observed. One easily underestimates the great demands made on the students by even the simplest astronomical concepts. Frequently, one thinks that the students have well understood a number of facts based on the drawings on paper, but when they are presented directly with the phenomena they can hardly find their way.

Take, for example, the illumination of the moon. The students usually learn the origin of the phases of the moon from a drawing (Fig. 21). One tells them: If sun and moon are in opposition, that is, if they rise and set twelve hours apart, it is full moon. The full moon therefore rises in the evening and sets in the morning. If sun and moon are in conjunction, if they rise and set at the same time, it is new moon. If, however, they are 90° apart in such a way that the moon rises at noon and sets at midnight, one only sees one-half illuminated – the moon is waxing. In the opposite case, the other half is illuminated – the moon is waning. It rises at midnight and sets at noon. Now, after discussing all this, just try to take the students out one evening, show them the moon, and ask them: Where is the sun now? The moon might, for instance, look like Fig. 22.

Fig. 21

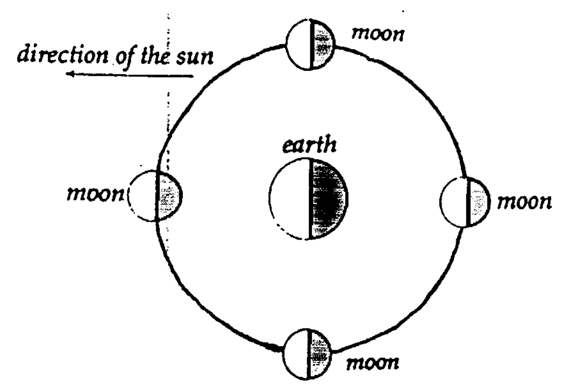


Fig. 22

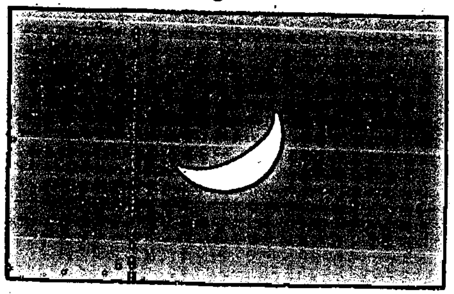
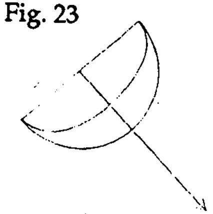


Fig. 23



It is a tremendous jump from the drawing of Fig. 21 to this question, and rarely will the students know what to make of it. One can make it much easier for them if one uses the opposite path. One illuminates a sphere, say the globe, with the projector and arranges it so that the students can see at one time more of the illuminated half, at another time more of the half lying in shadow. One shows them that the boundary between light and shadow, which is always a circle, appears at one time as a circle, at another time as a narrow ellipse. One guides them to derive from the visible part of the illuminated half the position of the entire illuminated half, and from it the direction of the light source, so that they will find from Fig. 22 the direction of the light source as the direction of the arrow in Fig. 23. This arrow does not lie in the plane of the drawing, but must be thought of as pointing away from the viewer. Now one can turn to the observation of the moon itself by saying that on the moon, too, one-half is always illuminated, the other is in shadow. As we saw with the illuminated sphere, now too we can determine the position of the illuminated half from its visible portion and from that the direction of the source of light. At this point one can explain to the students that the moon at night can never appear as it is often drawn in children's books, namely peeking into the window from above as in Fig. 24, because this means that the sun would have to be in the direction of the arrow in Fig. 25; the sun would be high above the horizon, it would be bright daylight. The teacher will further indicate that after sunset the moon would look something

Fig. 24

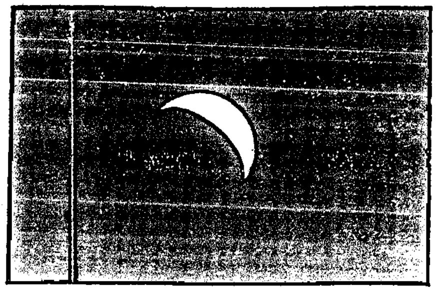
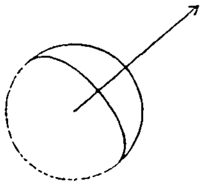


Fig. 25



like Fig. 26. Fig. 27 indicates the direction of the sun, where again the arrow must be thought of as pointing away from the viewer.

Fig. 26

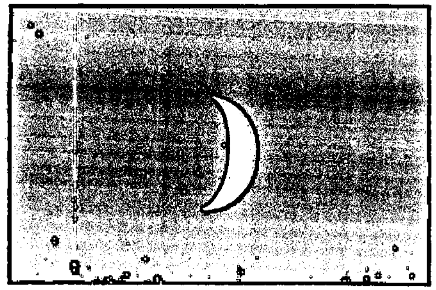
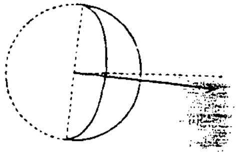


Fig. 27



On the other hand, if we see the moon a short time before sunrise, it looks like Fig. 28, where the direction of the sun is indicated by Fig. 29.

Fig. 28

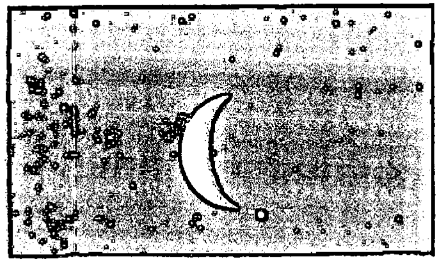
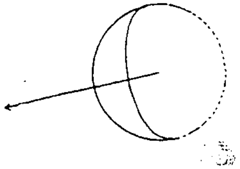


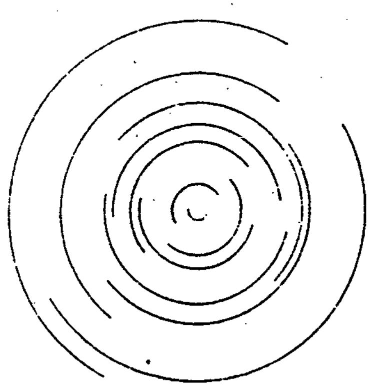
Fig. 29



Once the students have been led to visualize the position of the sun from the appearance of the moon, we can complete the lesson by showing how to tell time by the moon by estimating how far the sun has sunk below the horizon.

We can then take up the apparent circular motion of sun, moon, and stars due to the rotation of the earth. I have found many students who could recite Kepler's Laws very well, but were very astonished when we discussed the fact that the constellations of the fixed stars also rise in the east and set in the west, that in fact they appear to traverse the sky just as quickly as the sun and moon. In order to strengthen this point, it is even a good idea at first to emphasize that sun, moon and stars traverse the sky at the same rate. Only when this is understood should one describe how the sun lags a little, so that in the course of a year it has lagged a full circle. If one discusses this right at the start, the concepts will be confused, and the degree of this lag will be felt to be much too great. The concept of this motion of the stars can be made more alive if one mentions that astronomical observatories are equipped with a special clockwork mechanism so that the telescope automatically follows the stars, and also that the picture of a photographic plate exposed to the stars for a night shows a lot of circles. Fig. 30 represents a twelve-hour exposure toward the pole star.

Fig. 30



Only when the students have heard all this will they easily understand that the moon, if it is high in the sky in the evening, must set about midnight, so that the views of the moon in Figs. 31 and 32 cannot appear in one and the same night but must be

Fig. 31

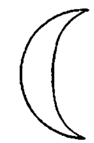
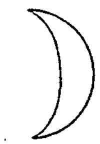


Fig. 32



assigned to two phases. Only now can one start discussing the apparent motion of the moon and can show how its lag causes the sequence: waxing moon (rises at noon), full moon (rises in the evening), waning moon (rises at midnight), and new moon (rises in the morning). The time of the synodic revolution does not have to be given abstractly as 29 1/2 days; one could, for instance, take a calendar, write down the dates of the full moons for the current year, and let the students compute the differences in days; in this way they themselves can find the period of the synodic revolution of the moon as actually experienced. For the year 1921, the results are as follows:

Full Moons	Intervals
January 23	29
February 21	30
March 23	29
April 21	29
May 20	30
June 19	30
July 19	29
August 17	29
September 15	30
October 15	30
November 14	30
December 14	

There are 5 intervals of 29 days and 6 intervals of 30 days, resulting in a synodic period of 29 1/2 days.

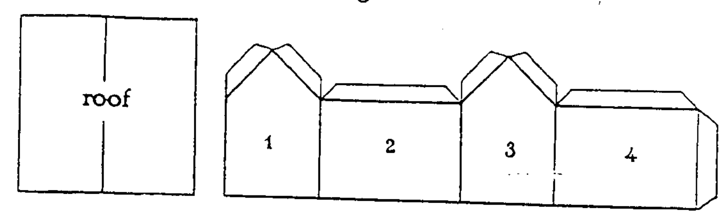
Another important point is always to present the big picture. Thus, in discussing the sun, the teacher should not only talk about corona, solar flares, etc., but immediately add the importance of the sun and its apparent motion for our earth. This is the occasion to expound on the statement of physics: "All energy comes form the sun."[7] One shows, for instance, how the sun is necessary for all motion on earth. The students themselves are asked to name different moving objects, and then the teacher explains how coal and oil are based on organic life, how animals and also human beings need plants, and how plants are unimaginable without sunlight. Here one can recall how potato shoots in dark cellars are white, never green, and how quickly they die. In regard to electric trains, the teacher has to show that the current in electric power plants is obtained by water power, and how it is the sun again which by its heat evaporates the water and raises it to the mountains; after falling as rain, it runs as brooks and streams down into the valleys and thus puts the generators into motion. The wind too is created by unequal warming of the air by the sun. Then one can mention that organic life would also be impossible under constant radiation by the sun, how the plants could not breathe, and how the daily rotation of the earth prevents one side of the earth being constantly in the light and the other side in the shade. This latter possibility would be the case if the earth and the sun were in the same relation as the moon and the earth (the moon always shows the same face to the earth).

Finally, it is especially important in the field of astronomy to tell the students not only what has been made known in the course of time about the sun, planets and fixed stars through scientific investigation, or what is based on still uncertain hypotheses, but to show how many unsolved problems remain, how many assumptions are still quite uncertain, to impart to the students the feeling of wonder that is the source of all philosophy and scientific striving.

One can proceed similarly with the quadrilateral, etc. A further step is the joining of plane figures to create solids. It is best to have the students draw the patterns of the several solids, including the tabs for pasting, then to have the students cut them out, score them, and paste them together. Here, the teacher can guide the students when making, say, a cube, to join squares with 3 cm sides to each other without first saying what the resulting solid will be. In this way, the students will be steadily occupied with this problem while drawing. They will be considering how it is going to fit together and will look forward to the final result of what they have been imagining.

If in this way one has constructed the most important solids: cubes, rectangular blocks, prisms, pyramids, cylinders and a cone, it is extremely stimulating to make a little house in the same way, e.g. in its simplest form:

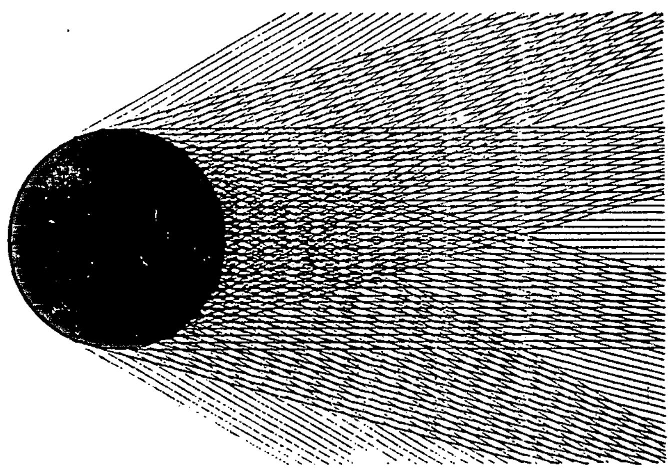
Fig. 37



This is something that works on the students and which they themselves will be stimulated to do at home in a variety of ways. Such undertakings already harbor several rules, for instance, the equality of planes 2 and 4. The students find the rules about regular figures themselves if, for instance, among the many triangles which they draw, the teacher develops one by drawing 60° angles at both sides of the base and then asks the students to measure the sides. It then becomes a happy experience for the students that all three are the same, a perception which they will not soon forget.* Similarly, one checks the angle at the apex of the triangle, so that it is also evident that all the angles are equal. One can proceed in the same way with the square and the regular hexagon.

It is important that the teacher does not present problems to the students without first carefully developing at a more elementary stage the abilities necessary for their solution. One finds that most students starting projective geometry have great difficulties with the spatial visualization of anything connected with projecting and projections. These requirements of visualization are often expected from the students for the first time at this point. A useful tool as preparation at a more elementary stage is the study of shadows,** especially the study of cast shadows.

Fig. 38

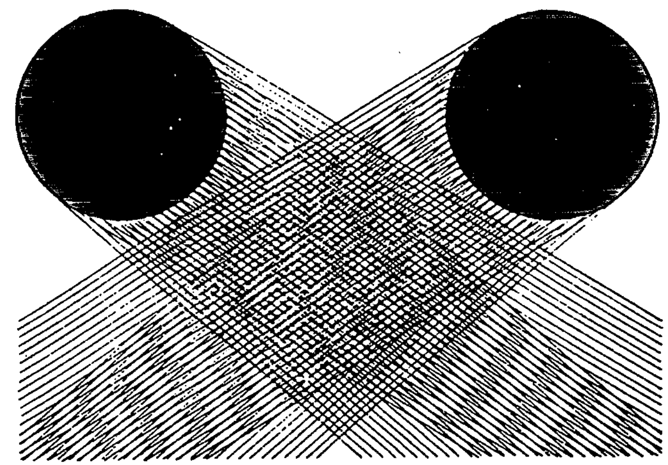


* This may by no means be presented as a proof but it prepares the questioning which will then be answered by the proof.

** This is scheduled by Rudolf Steiner in the curriculum of the Waldorf School for the sixth grade.

In this case we do not have to limit ourselves to *one* light source but can perhaps develop the following configuration: A cylinder illuminated from various sides casts shadows on its base plane (Fig. 38). Such problems create fewer difficulties than one might think. It is a particularly joyful experience to observe such a variety of patterns. One can also proceed to the crossing of shadows (Fig. 39).

Fig. 39



On these occasions, a number of exercises can be presented which will be of great importance in subsequent instruction, e.g. the transition from circle to ellipse. Consider a semi-circle standing vertically on the plane of the paper illuminated by a set of vertically spaced light sources casting shadows on the plane of the paper (Fig. 40). To create a total picture from the sequence of lessons, it is good to discuss the same figures at different stages from different points of view. If earlier in treating different forms of triangles one has arrived at a pattern like that in Fig. 36 b, one can now arrive at a very similar pattern by assuming a triangle to be illuminated by a set of vertically spaced light sources casting shadows of various lengths on the plane of the paper. In this way, the transformation from one triangle to another discussed earlier now becomes reality (Fig. 41).

Fig. 41

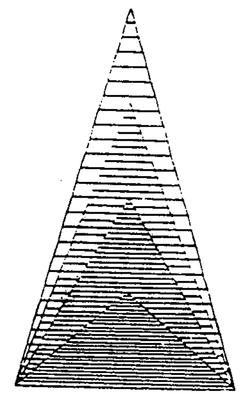
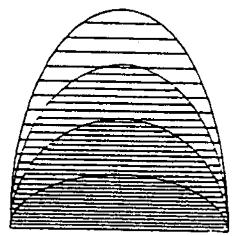


Fig. 40



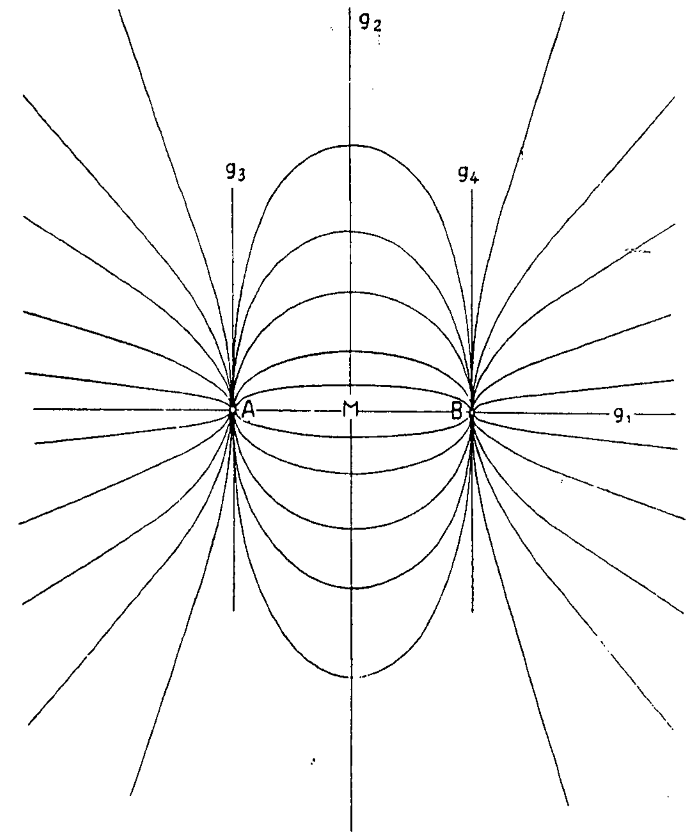
It must be a principle of teaching that one's purpose should not be to attempt to cover a particular material, but by using the material to develop a particular ability in the students. In studying the geometry of curves, if one considers primarily their variability, this could be the best method of awakening a certain flexibility of thought. For this purpose, a number of examples of second-order curves, as taught in the upper grades, will be presented. It will be shown what transformations are possible when fixing various elements. Here, one will also have the opportunity to introduce vividly some concepts of modern geometry.

1.Fix the end points of an axis (*A* and *B*) and move the foci (Fig. 42).

If the foci are near *A* and *B* but inside the line segment $\overline{AB}$ one obtains very narrow ellipses. The closer the foci converge on the center *M*, the closer the ellipse approaches a circle. It is completely transformed into a circle when the foci meet at *M*. If one continues this transformation, the foci diverge on the line g_2, and the ellipses stretch vertically until the portion remaining on the paper ap-

proaches the lines g_3 and g_4. If, on the other hand, the foci are very

Fig. 42



close to *A* and *B* but outside the line segment $\overline{AB}$, one obtains the very narrow hyperbolas which deviate less from g_1 the closer the foci approach *A* and *B*. Moving the foci in the other direction, away from *A* and *B*, will cause the hyperbolas to get wider until they too approach the lines g_3 and g_4.

The continuous transition from the narrow ellipses, whose limiting case is the line segment $\overline{AB}$, to the circle, further to the long vertical ellipses, to the parallel lines g_3 and g_4, to the wide hyperbolas and from them to the narrow ones, finally reaching the limiting line g_1 outside $\overline{AB}$, represents in summary a transition from the inner line to the outer one.

The second consideration in this continuous transition is the motion of the foci. First they move from *A* and *B* inward to *M*, then on the orthogonal line further and further out. Then comes a jump to line g_1 and a gradual approach to *A* and *B* until at *A* and *B* they return to the starting point.

One can hardly imagine a better exercise than this metamorphosis to develop flexibility of thought. Of course, it must be presented in a variety of ways.

2. Fix the foci (F_1, F_2); allow the axis to vary (Fig. 43).

If the end points of one axis lie within the line segment $\overline{F_1F_2}$, one obtains hyperbolas which approach the line g_2 as *A* and *B* approach M, but get narrower and approach the portion of line g_1 outside segment $\overline{F_1F_2}$ as *A* and *B* approach F_1 and F_2. If, on the other hand, *A* and *B* lie outside F_1 and F_2, one gets ellipses which approach a circle as *A* and *B* move outwards. In the contrary case, however, if *A* and *B* coincide with F_1 and F_2 the ellipses are transformed into the line segment $\overline{F_1F_2}$. Here, one can show that, as *A* and *B* move out from *M*, the figures are transformed from the line g_2 via the hyperbolas to the outer part of g_1, then suddenly to the inner part of g_1, and finally via the ellipses to the circle.[8]

3. Fix the four points (*A*, *B*, *C*, *D*); allow the foci to vary (Fig. 44).

Here, too, one can show how a continuous transformation from one curve to another is possible. If one starts, say, with the

circle, one can move the foci outward along line g_1 and arrive at longer and longer ellipses which approach the two parallel lines g_5 and g_6. The points of intersection of the ellipses and g_2 approach points *N* and *P* from the outside. If one now crosses points *N* and *P* to reach the interior line segment $\overline{NP}$ one obtains hyperbolas which finally reach their limit in lines g_7 and g_8. But one can reach these lines also in another way. Starting again from the circle, if one moves the foci outward along line g_2, one gets by way of longer and longer ellipses to g_3 and g_4. The points of intersection with g_1 now approach *Q* and *R* from the outside. Crossing *Q* and *R*, one again gets hyperbolas which finally run into the diagonals g_7 and g_8.

Fig. 43

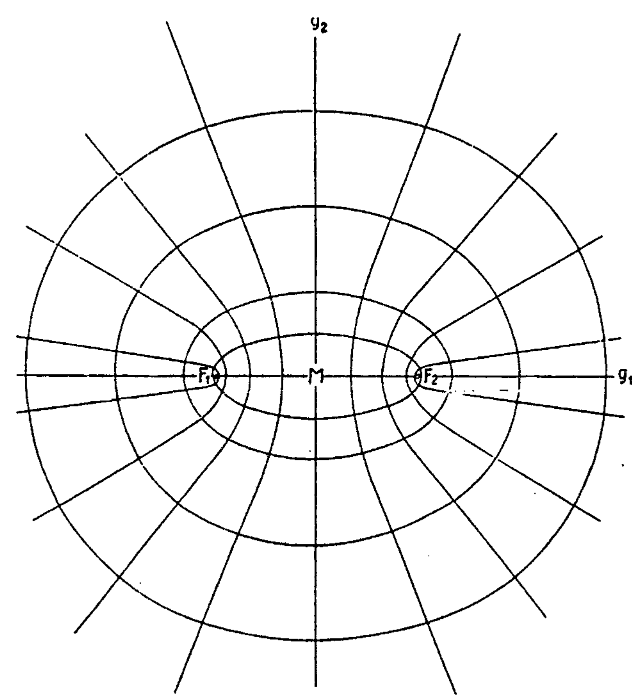
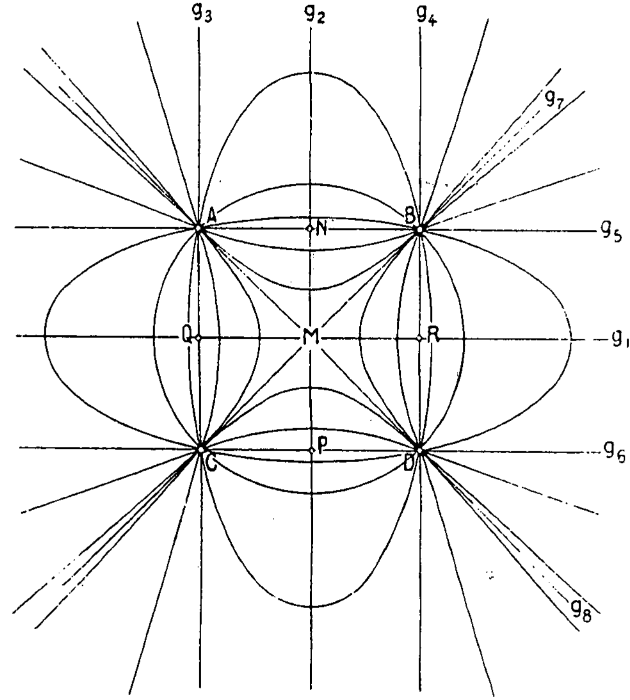


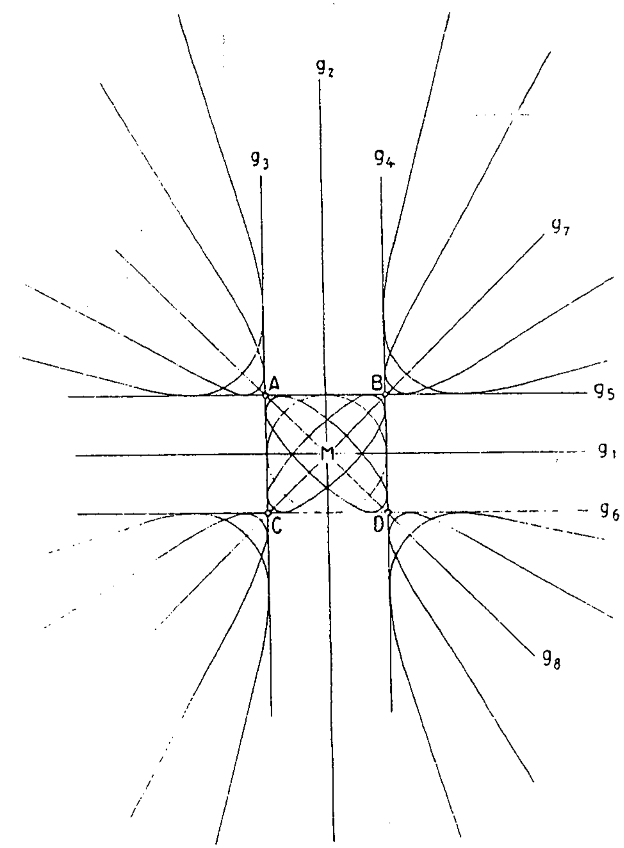
Fig. 44



4. Fix four tangents (g_3, g_4, g_5, g_6) (Fig. 45).

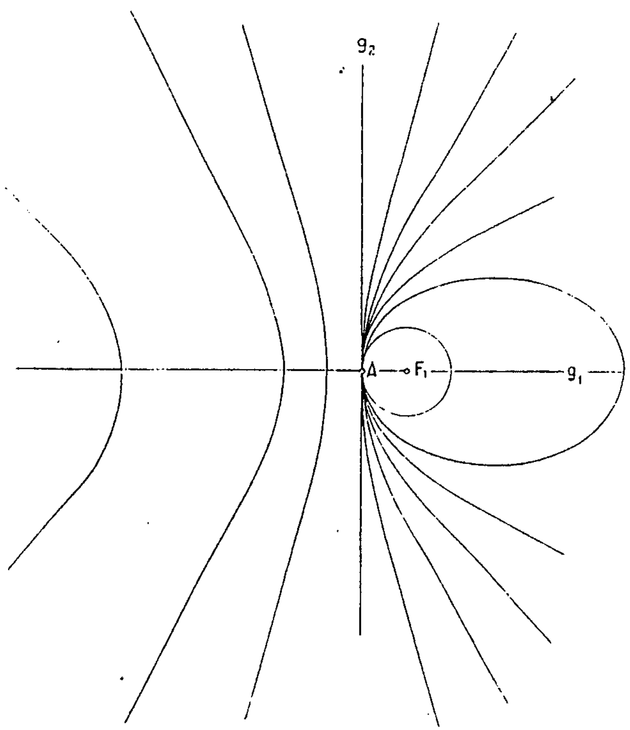
Starting with the circle and keeping g_3, g_4, g_5, g_6 as tangents, there are two ways of obtaining ellipses. One possibility is through constantly narrower ellipses to line segment $\overline{AD}$. The intersection of these ellipses with g_8 approaches the points A and D and finally coincides with them. If one continues beyond A and D one transforms to hyperbolas which become wider the further one continues in this manner. The second possibility is analogous, merely substituting g_7 for g_8 and $\overline{CB}$ for $\overline{AD}$

Fig. 45



5. Fix one focus, F_1, and one axis end point, A (Fig. 46).

Fig. 46

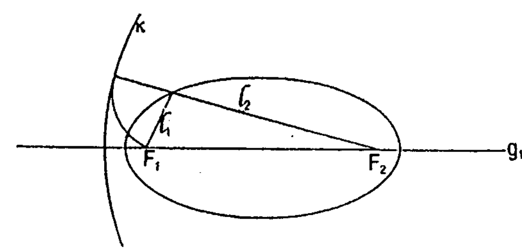


This transformation shows the transition via the parabola. As long as the foci or the axis end points move equally outward, one retains twofold symmetry. One obtains ellipses and hyperbolas. If,

however, one fixes one focus and its axis end point, and moves the other focus and its axis end point further and further to the right, one obtains ellipses which approach a parabola. If one continues the curvature at A in the same sense, the focus, which moved further and further to the right, reappears on the left, the parabola is transformed to a hyperbola whose branches become flatter and approach each other until they finally transform into line g_2. This is a good demonstration of a concept of modern geometry, the single infinitely far point of a straight line (g_1).

That a parabola really appears in the transition from ellipse to hyperbola is easily proved by noting the change of the basic circle (center = F_2, radius = major axis) (Fig. 47).[9]

Fig. 47



If F_2 moves to infinity, l_2 becomes parallel to g_1. The basic circle is transformed to a straight line, and the definition of the ellipse transforms into that of the parabola. It is similar with the hyperbola (Fig. 48):

If F_2 moves further and further to the left, l_2 becomes more and more parallel to g_1 and the basic circle approaches the same straight line from the left, indicating again the transition to the parabola.[10]

To make the foregoing more vivid, one can also show the transitions from one curve to the other by the intersection of a plane with a double cone, changing the position of the intersecting plane in an appropriate manner.

Fig. 48

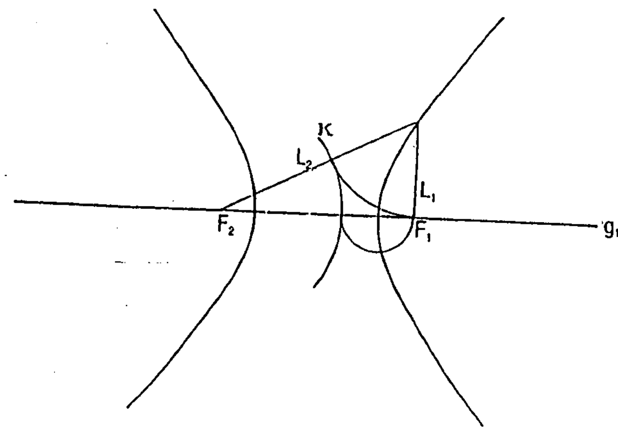
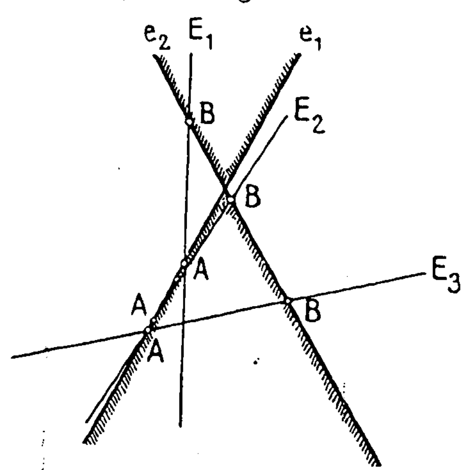


Fig. 49 represents this for Case 1. Lines e_1 and e_2 are the generators of a cone shown in section, E_1, E_2, E_3 the projections of the

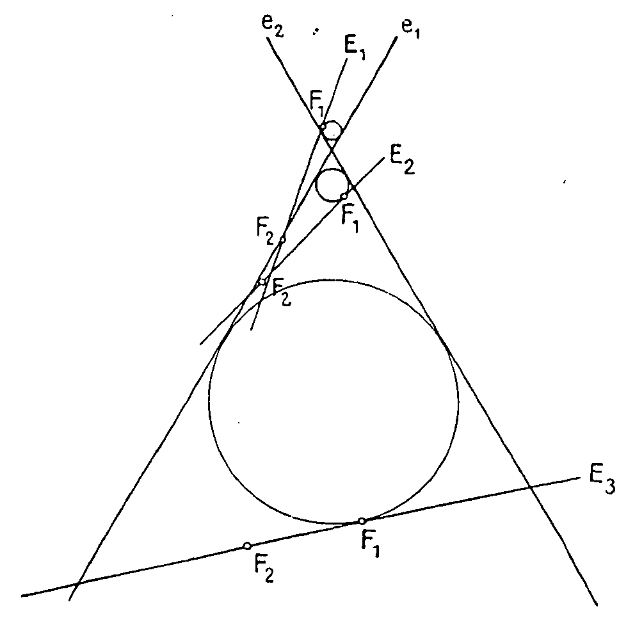
Fig. 49



intersecting planes. The movement of the intersecting planes must proceed in such a manner that the distance $\overline{AB}$ between the lines e_1 and e_2 remains constant.

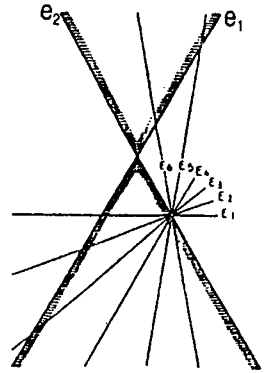
For Case 2, the movement of the intersecting plane must take place as in Fig. 50, which is again to be taken as a cross section of a double cone with the generators e_1 and e_2. The transition of E_1, E_2, E_3, etc. must be performed in such a way as to keep the distance $\overline{F_1F_2}$ constant, as indicated in the drawing using Dendelin's Theorem.[11]

Fig. 50



The transition via the parabola can be obtained by a rotation of the intersecting plane. The rotation shown in Fig. 51 indicates this most clearly, but does not correspond exactly to Case 5, since the distance $\overline{F_1A}$ is not constant in this case. For that, a motion of the point of rotation would be necessary.

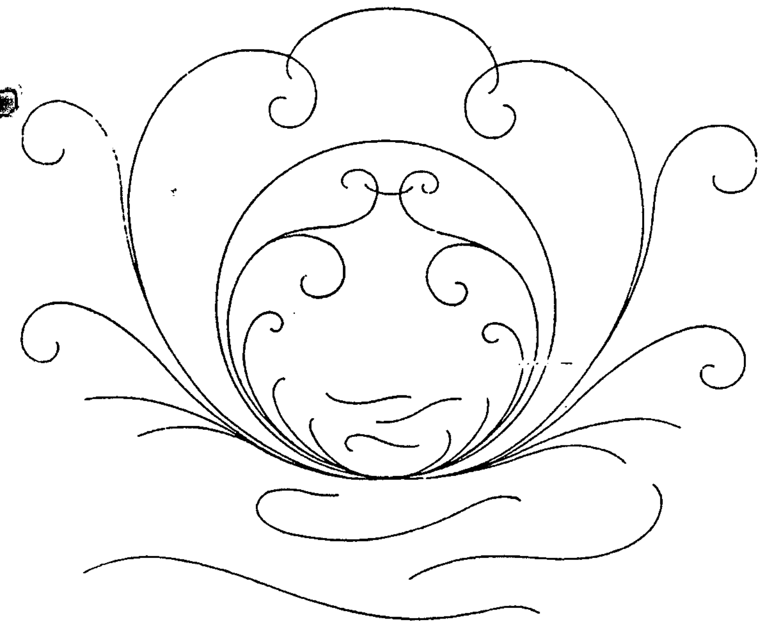
Fig. 51



In conclusion, it should be pointed out how much can be gained in the teaching of geometry if one tries to find examples and stimulation in many areas. For instance, symmetry can be demonstrated very beautifully in plants, e.g. the violet, where not only is the flower symmetrical, but the spur extends in the plane of symmetry, and the entire stem carrying the flower bends only in the plane of symmetry.

Finally, instruction in drawing is a good preparation for understanding geometry. Just as one can develop the feeling for symmetry by drawing symmetrical forms, so one can go beyond ordinary reflection to reflection in a circle and perhaps present an exercise like this: Given a circle, let the students draw forms in such a way that to every form *inside* the circle there is a corresponding form *outside*, so that a form which must fit itself into the interior of the circle can expand outside of it. Here, one does not have to do a pedantic construction, but one can let a healthy feeling for form carry out this metamorphosis. It is obvious how valuable this can be. The understanding for the relation of points inside and outside the circle, a fundamental concept in function theory, will be awakened in this vivid way, and an important problem will arise: where does the point corresponding to the center lie? The answer is: far distant in any direction. Fig. 52 presents an example of such an arrangement.*

Fig. 52



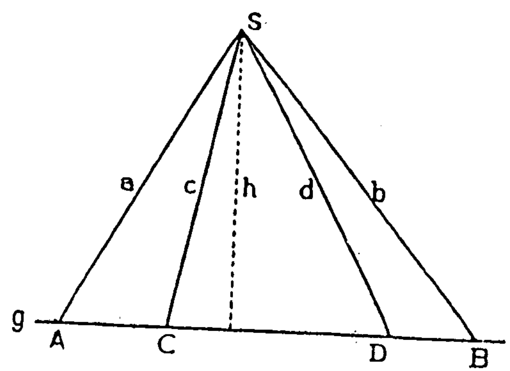
* Further material in this direction may be found in the author's *Geometrie in Bildern* in the folder *"Bilder aus verschiedenen Gebieten der Geometrie"*.

Besides the principles already mentioned, it is especially important with mathematics to shape everything we present in such a way that it is thoroughly understood by the students. A mathematical proof should not merely be a logical chain that forces the students to accept the result as proven, but when they try to carry out the proof themselves they will have to depend at least at some points on their memory. Instead, a proof must be developed in such a way that the student can always rediscover it. This is not only true at the high school level, but also in college. Examples will follow to show what is meant.

First we have chosen the invariance of the cross ratio in projective transformations. The usual proof goes as follows:

To prove the invariance of the cross ratio $(ABCD) = \frac{\overline{AC}}{\overline{AD}} : \frac{\overline{BC}}{\overline{BD}}$, one represents the areas of the triangles *ACS, ADS, BCS, BDS* in two ways (Fig. 53):

Fig. 53



1. As half the product of base and height:

$$\Delta ACS = \overline{AC} \cdot \frac{h}{2}$$

$$\Delta ADS = \overline{AD} \cdot \frac{h}{2}$$

$$\Delta BCS = \overline{BC} \cdot \frac{h}{2}$$

$$\Delta BDS = \overline{BD} \cdot \frac{h}{2}$$

2. As half the product of two sides and the sine of the included angle:

$$\Delta ACS = ½ \cdot a \cdot c \cdot \sin(ac)$$
$$\Delta ADS = ½ \cdot a \cdot d \cdot \sin(ad)$$
$$\Delta BCS = ½ \cdot b \cdot c \cdot \sin(bc)$$
$$\Delta BDS = ½ \cdot b \cdot d \cdot \sin(bd)$$

If one now takes the cross ratio of the areas in both expressions and sets these equal, one obtains:

1. $$\frac{\Delta ACS}{\Delta ADS} : \frac{\Delta BCS}{\Delta BDS} = \frac{\overline{AC} \cdot h/2}{\overline{AD} \cdot h/2} : \frac{\overline{BC} \cdot h/2}{\overline{BD} \cdot h/2} = \frac{\overline{AC}}{\overline{AD}} : \frac{\overline{BC}}{\overline{BD}},$$

2. $$\frac{\Delta ACS}{\Delta ADS} : \frac{\Delta BCS}{\Delta BDS} = \frac{½a \cdot c \cdot \sin(ac)}{½a \cdot d \cdot \sin(ad)} : \frac{½b \cdot c \cdot \sin(bc)}{½b \cdot d \cdot \sin(bd)}$$
$$= \frac{\sin(ac)}{\sin(ad)} : \frac{\sin(bc)}{\sin(bd)}$$

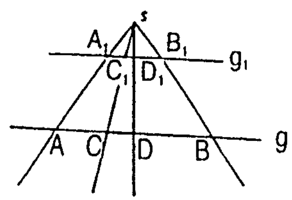
Equating, $$\frac{\overline{AC}}{\overline{AD}} : \frac{\overline{BC}}{\overline{BD}} = \frac{\sin(ac)}{\sin(ad)} : \frac{\sin(bc)}{\sin(bd)}$$

Now, on the left of the equation we have the cross ratio that is to be proved constant regardless of the position of line *g*. On the right side is an expression containing only sines of the angles between the four projection rays, therefore independent of the position of *g*. This proves the theorem.

Simple as this proof seems to be, one would hardly, without appealing to memory, think of using the areas of these triangles and of representing them in two different ways in order to prove the invariance of the cross ratio.

In contrast, a second proof will now be presented in which nothing shall be used which does not organically flow from the problem itself. First, one considers that due to the similarity of the triangles any transformation in which g' is parallel to g, as in Fig. 54, will shorten all lines equally, so that not only the cross ratio, but all ratios are invariant. The proof of the invariance of the cross ratio depends therefore only on the rotation. We will therefore draw line g' rotated with respect to g.

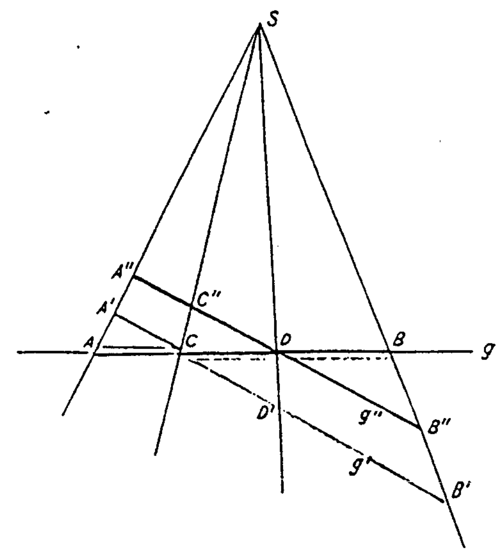
Fig. 54



The question is, about what point we should rotate in order to obtain the simplest possible arrangement. In the cross ratio $(ABCD) = \frac{\overline{AC}}{\overline{AD}} : \frac{\overline{BC}}{\overline{BD}}$ there is no special point of rotation since the first ratio $\frac{\overline{AC}}{\overline{AD}}$ is concerned with the distances form A to C and D, the second ratio with those from B to C and D. Rather than choosing *one* point of rotation, it is more natural to choose two, say C and D (Fig. 55). One can see immediately from the similarity of triangles ACA' and ADA'' that the ratio $\frac{\overline{AC}}{\overline{AD}}$ is equal to the ratio $\frac{\overline{A'C}}{\overline{A''D}}$, and similarly that the ratio $\frac{\overline{BC}}{\overline{BD}}$ reappears as $\frac{\overline{B'C}}{\overline{B''D}}$. But now the magnitudes under consideration have been apportioned to two different lines, in such a manner that the numerators of both

ratios, $\overline{A'C}$ and $\overline{B'C}$, lie on g', while the denominators $\overline{A''D}$ and $\overline{B''D}$, lie on g''. But if one projects the distances of line g'' from S to the parallel line g', the denominators are increased by the same proportion, so that each ratio changes, but the cross ratio remains invariant. This proves, therefore, that $(ABCD) = (A'B'CD')$, where g' goes through C but can have any arbitrary slope. Since the invariance of the cross ratio under any projection onto a parallel line is obvious, the general proof is complete.

Fig. 55



If such proofs are often more wordy, one can nonetheless, if one has followed the thought process, see the entire proof by just looking at the figure, and it creates not only the assurance that the cross ratio remains intact, but one sees also how the theorem came into being.

To furnish another example which is not connected with geometry, let us consider the constant e. It is often introduced as the following limiting value: $\lim_{n \to \infty}\left(1+\frac{1}{n}\right)^n = e$. While it is usually demonstrated that this series converges, it leaves the student with the question of why the limit of just this particular series plays such a fundamental role in mathematics and physics. We have to ask ourselves: could the limit of any other converging series not be accepted as the base of the natural logarithm?

If we now look for the property of the number e which gives it its importance in mathematics and physics we find that its appearance almost always derives from the differential equation $\frac{dy}{dt} = y$, where y is a quantity growing with time whose law of growth is represented by this differential equation.. The latter says that the speed of growth dy/dt is always equal to the value reached by y itself, in other words, that y grows faster the greater it becomes. Now we can show how this type of growth results in the series we just presented by letting the growth occur, not continuously, but in discrete jumps. Thus, instead of $dy = y \cdot dt$, one now takes $\Delta y = y \cdot \Delta t$, where Δy and Δt are finite changes in y and t.

1. $\Delta t = 1$, starting point for $t = 0$ is $y = 1$.

$t = 0, y = 1$

$t = 1, y = 1 + 1 = 2$

$t = 2, y = 2 + 2 = 4 = 2^2$

$t = 3, y = 4 + 4 = 8 = 2^3$

$t = 4, y = 8 + 8 = 16 = 2^4$

....

$t = T, y = 2^T$

Thus, y in this case grows as a power of 2.

If we now make the jumps at half-second intervals we obtain

2. $\Delta t = ½$, for $t = 0 \quad y = 1$

$t = ½ \quad y = 1 + ½$

$t = 1 \quad y = (1 + ½) + ½\,(1 + ½) = (1 + ½)^2$

$t = 3/2 \quad y = (1 + ½)^2 + ½\,(1 + ½)^2 = (1 + ½)^3$

$t = 2 \quad y = (1 + ½)^3 + ½\,(1 + ½)^3 = (1 + ½)^4$

....

$t = T \quad y = (1 + ½)^{2T} = [(1 + ½)^2]^T$

Now y grows to a power whose base is no longer 2, but $(1+1/2)^2$.

If finally we let the jumps happen after $1/n$ second, we get:

3. $\Delta t = 1/n$, for $t = 0 \quad y = 1$

$t = \frac{1}{n} \quad y = 1 + \frac{1}{n}$

$t = \frac{2}{n} \quad y = (1 + \frac{1}{n}) + \frac{1}{n}(1 + \frac{1}{n}) = (1 + \frac{1}{n})^2$

$t = \frac{3}{n} \quad y = (1 + \frac{1}{n})^2 + \frac{1}{n}(1 + \frac{1}{n})^2 = (1 + \frac{1}{n})^3$

....

$t = \frac{n}{n} = 1, \quad y = (1 + \frac{1}{n})^{n-1} + \frac{1}{n}(1 + \frac{1}{n})^{n-1} = (1 + \frac{1}{n})^n$

....

$t = T \quad y = (1 + \frac{1}{n})^{nT} = \left[(1 + \frac{1}{n})^n\right]^T$

Here, we already arrive at the power series whose base is $(1 + \frac{1}{n})^n$. The greater we let n get, the more jumps we make in unit time, the closer we get to the continuous case, so that the base of the power series corresponding to continuous growth is $\lim_{n \to \infty}\left(1+\frac{1}{n}\right)^n$.

With this derivation we have not presented an arbitrary series to the students, but have shown them how growth conditions

which frequently appear in the real world lead to this series. Eve
more frequent than the pure growth equation $dy/dt = y$ is th
equation $dy/dt = ay$, resulting by an analogous development in th
power $y = e^{at}$. Here, the differential equation requires that th
speed of growth is always in a certain constant ratio to the value o
y.[12] Every real situation in which growth is slower at the beginnin
than later, e.g. the amount of wood in a forest which increase
faster the bigger the trees are, is an example of this.[13] This type o
growth, which leads to t-th powers, can be contrasted with on
growing according to differential equations of the form $d^ny/dt^n =$ k
which result in n-th powers of t. This contrast in continuou
growth is analogous to the contrast between geometric and arith
metic series in stepwise growth.

In arithmetic series (for simplicity, a first-order series is cho
sen) such as the following:

$$a, a + d, a + 2d, a + 3d, \ldots, a + (n - 1)d, \ldots$$

the same magnitude is always added regardless of the valu
reached by the function itself. The essence of the geometric serie
consists in the fact that the growth depends on the magnitude o
the function itself, that it is always a multiple of that function, sa
ka. The series is then:

$$a, a + ka, (a + ka) + k(a + ka), [(a + ka) + k(a + ka)] + k[(a + ka) + k(a + ka)] \ldots.$$

$$= a, a(1 + k), a [(1 + k) + k (1 + k)], a\{[(1 + k) + k(1 + k)] + k [(1 + k) + k(1 + k)]\} \ldots..$$

$$= a, a (1 + k), a (1 + k)^2, a (1 + k)^3 \ldots.. a (1 + k)^{n-1}$$

and therefore geometric. In this way, one can vividly demonstrate
the difference between arithmetic and geometric series; this
demonstration can then be completed by providing practical
examples. The growth according to an arithmetic series, for
instance, is represented by the growth of velocity of a falling

sphere, where in every second the value of earth's gravity, g, is added, independent of the velocity reached. On the other hand, the growth of capital in a bank through interest is geometric, since here the growth is based on what is already there. The more capital, the more interest.

Conclusion

Finally, it should be emphasized that in the teaching profess more than in any other, it is important for the teacher to devote his strength and all his enthusiasm to the cause. The best curri lum is useless if the teacher's initiative does not let him find right approach to every lesson and to every sentence he utters. T initiative, created by the teacher's own enthusiasm, should considered the most important aspect of teaching. Yet the more requirements set by the state take away the teacher's own freedo and responsibility, the more this precious power for instructi will be lost, and pedantry will take its place. Regulations, matter how good, cannot help us in these difficult times. There only the unrestrained strength of the individual. We must put o trust in that.

Translator's Notes

his was, of course, written before the acceptance of plate tectonics ory.

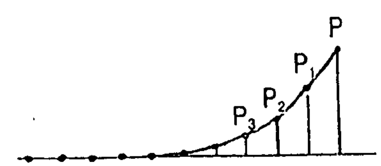
Another example, perhaps more familiar to today's students, is the dency of a bank of elevators to run together, arriving at the same floor he same time, unless artificially controlled by computerized schedul-

ome five years after this paper by Dr. Baravalle, Albert Einstein pre- ited another plausible explanation of the formation of meanders in his per "The Cause of the Formation of Meanders in the Courses of Rivers, d of the so-called Baer's Law", presented on January 7, 1926 to the ssian Academy of Science. A translation by Alan Harris appears in the ok A. Einstein, *Essays in Science*, Philosophical Library, NY, 1934. efly, and much simplified, an initial random small curvature in the river d will cause a centrifugal force on the water toward the outside of the rve. Due to friction on the bottom, the outward velocity caused by this ce will be less on the bottom than near the surface, resulting in a rotation nsverse to the direction of the stream, outward near the surface and ward near the bottom of the river. This in turn will cause sand and gravel be carried from the outside of the curve to the inside, causing the rvature to increase.

Of course, the teacher now would want to substitute the cab of a Diesel gine or the cockpit of an airplane.

The author unaccountably omits the gas pedal.

Fig. 12 clearly does not picture the initial transient if simple periodic otion of point B is assumed. The figure should look more like this:



. Since the discovery of nuclear energy, this statement is no longer true. f the sun's energy is nuclear energy, one could say: "All energy is nu- lear," but this presents the student with an abstraction of little immediate nterest. It would be much better to say: "Except for a tiny fraction pro- luced by nuclear energy, all energy on earth comes from the sun." One

might also discuss geothermal energy and point out that it, too, mı
nuclear or solar in origin.

8. It should be noted that with the foci separated by a fixed distance a
will not be reached, only approached; as the axis $\overline{AB}$ approaches ir
length, the ellipse will approach the shape of a circle with infinite di
ter.

9. The concept of the basic circle may not be familiar. As just defined,
the ellipse is defined as the locus of points the sum of whose dist
from the two foci is a constant equal to the major axis, a circle K (Fi
can be drawn with center at F_2 and radius $l_2 + l_1$ as shown i
construction. Similarly, in Fig. 48 circle K has a radius $l_2 - l_1$, correspon
to the definition of the hyperbola. With axis end point A and focus F_1 f
the intersection of the basic circle with the major axis is also fixe
distance $\overline{AF_1}$ from A on the opposite side from F_1. For ellipses, K
curve to the right (Fig. 47); for hyperbolas, it will curve to the left (Fig
for the parabola, K will degenerate to a straight line.

10. A similar transition can be obtained by fixing one focus and
directrix and varying the eccentricity.

11. Dendelin's Theorem states that, for a conic section, a sphere inscr
in the cone and tangent to the intersecting plane will be tangent to
plane at one focus of the conic section.

12. If t is in units of time, the "pure growth equation" dy/dt =
dimensionally inconsistent; it only makes sense if t is non-dimensio
Therefore, in actual applications of physics, we always find the equa
in the form $dy/dt = ay$, where a has the dimensions of time^{-1}. Of cou
any equation of the form $dy/dt = ay$ can be converted to the form dy/dt
by letting $t' = at$.

13. This statement is certainly an exaggeration. There are many oc
rences in nature which grow slower at the beginning than later and do
vary exponentially; e.g. the distance covered by a body falling under
force of gravity which varies with the square of the time. In reality, wh
much growth in nature starts exponentially, none continues in this fash
since nothing can grow without limits; there is always a point of satu
tion.

ts by Hermann v. Baravalle in English translation

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) (Waldorf School Monographs)

oduction to the Astronomical Phenomena,
th Astronomical Almanac 1943

spective Drawing (1968) (Waldorf School Monographs)

lolf Steiner as Educator (1952)

e Teaching of Arithmetic and the Waldorf School Plan (1967)
aldorf School Monographs)

e Waldorf School Plan

e Astronomical Phenomena of the Year 1941

e International Waldorf School Movement (1963)
aldorf School Monographs)